Journal of Statistical Physics, Vol. 106, Nos. 1/2, January 2002 (© 2002)

A Note on the Metastability of the Ising Model: The Alternate Updating Case

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Received May 7, 2001; accepted August 30, 2001

We study the metastable behavior of the two-dimensional Ising model in the case of an alternate updating rule: parallel updating of spins on the even (odd) sublattice are permitted at even (odd) times. We show that although the dynamics is different from the Glauber serial case the typical exit path from the metastable phase remains the same.

KEY WORDS: Spin models; stochastic dynamics; metastability; critical droplets.

There have been many studies of the problem of metastability in the Ising model, i.e., the time evolution of a system initially in the all minus phase in a small positive external magnetic field. The case of Glauber-Metropolis serial dynamics has been discussed in ref. 1, where it was proved that the exit from the metastable phase is achieved via the nucleation of a critical droplet and that the life time of the metastable phase depends only on the energy of such a critical droplet. The proof, which uses the pathwise approach introduced in ref. 2, is based on the analysis of the tendency of droplets of pluses in a background of minuses to shrink or to grow. In ref. 3 it was shown that the behavior of such droplets does not change if the spin flip rates are modified.

In this note we consider the case of parallel dynamics in which spins on the even (odd) sublattice are simultaneously updated. More precisely: we consider the two-dimensional Ising model on a finite rectangle $\Lambda \subset \mathbb{Z}^2$ with even side length and periodic boundary conditions; for any $x \in \Lambda$,

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$\sigma(x) = \pm 1$ denotes the spin variable at that site and to any configuration $\sigma \in \Omega := \{-1, +1\}^{\Lambda}$ we associate the energy

$$H(\sigma) := -\frac{J}{2} \sum_{\langle x, y \rangle} \sigma(x)\, \sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x), \tag{1}$$

where $J \gg h > 0$. The equilibrium is described by the Gibbs measure $\nu(\sigma) := \exp\{-\beta H(\sigma)\}/Z$ with β, the inverse of the temperature, a positive real number and $Z := \sum_{\eta \in \Omega} \exp\{-\beta H(\eta)\}$ the partition function.

We define the following updating rule: we partition Λ into its even and odd sublattices, $\Lambda = \Lambda^e \cup \Lambda^o$, with $\Lambda^{e(o)} := \{x = (x_1, x_2) \in \Lambda : x_1 + x_2$ is even (odd)$\}$ and we denote by $\Xi^{e(o)} := \{\{x_1,..., x_m\} : 1 \leqslant m \leqslant |\Lambda|/2, x_i \in \Lambda^{e(o)}$ for all $i = 1,..., m\}$ the set of all possible collections of sites in $\Lambda^{e(o)}$. Given $I \in \Xi^{e(o)}$ we denote by σ^I the configuration obtained by flipping in σ the $|I|$ spins associated to the sites in I. Let $\mu^{e(o)}$ be a probability measure on $\Xi^{e(o)}$ such that $\mu^{e(o)}(I) > 0$ for any $I \in \Xi^{e(o)}$ and $t = 0, 1,...$ an integer variable. Finally, at any even (odd) time t a set $I \in \Xi^{e(o)}$ is chosen at random with probability $\mu^{e(o)}(I)$ and the $|I|$ spins associated to sites in I are flipped with probability $\exp\{-\beta[H(\sigma^I) - H(\sigma)]^+\}$, where σ is the configuration at time t. In other words we can say that the evolution of model (1) is described by a Markov chain σ_t with transition probability $P(\sigma_{t+1} = \eta \mid \sigma_t = \xi) = p_\alpha(\xi, \eta)$, with $\alpha = e, o$ depending on t even or odd, where for any $\eta \neq \xi$

$$p_\alpha(\xi, \eta) := \begin{cases} \mu^\alpha(I) \exp\{-\beta[H(\xi^I) - H(\xi)]^+\} & \text{if } \exists I \in \Xi^\alpha \text{ such that } \eta = \xi^I \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and $p_\alpha(\xi, \xi) := 1 - \sum_{\eta \neq \xi} p_\alpha(\xi, \eta)$. We remark that this dynamics satisfies the detailed balance condition with respect to the Gibbs measure, hence this is its unique invariant measure.

Our dynamics lives in the same configuration space with the same energy landscape as the standard single spin flip Metropolis algorithm, but many more connections between different configurations have been opened.[4] In other words we allow the system to perform jumps forbidden in the standard serial case. To be more precise the notions of "communicating states" (allowed jumps) and, hence, of connected sets have been changed: we say that $\sigma, \eta \in \Omega$ are *e(o)-nearest neighbors* (e(o)-nn) iff $\exists I \in \Xi^{e(o)}$ such that $\eta = \sigma^I$; two configurations are nearest neighbors if they are e-nn or o-nn, in other words any two configurations when differing only on the evens or odds sides are neighbors. A set $\mathscr{A} \subset \Omega$ is connected iff for any $\sigma, \eta \in \mathscr{A}$ there exists a sequence $\sigma_0,..., \sigma_n$ of configuration such that: $\sigma_0 = \sigma$, $\sigma_n = \eta$ and for any $i = 0,..., n-1$, depending on i even or odd, σ_i and σ_{i+1} are e(o)-nn.

The fact that new jumps have been allowed could, a priori, modify the metastable character of the dynamics and eventually destroy it. In the most extreme case single spin flip paths containing steps against the energy drift can be replaced by direct energy favored jumps. Indeed, let us consider $\sigma \in \Omega$ and $I = \{x_i\colon 1 \leqslant i \leqslant n\} \in \Xi^{e(o)}$, it is easy to show that

$$H(\sigma^I) - H(\sigma) = \sum_{i=1}^{n} [H(\sigma^{x_i}) - H(\sigma)]; \tag{3}$$

one just has to remark that for any $i, j = 1,..., n$ and $i \neq j$ the two sites x_i and x_j are not nearest neighbors. In other words (3) means that from the point of view of the energy a parallel event can be substituted by a suitable sequence of single spin flip events. Now, one can easily choose σ and $I = \{x_i\colon 1 \leqslant i \leqslant n\}$ such that $H(\sigma^I) \leqslant H(\sigma)$ and $H(\sigma^{x_i}) - H(\sigma) > 0$ for some $i = 1,..., n$.

Nevertheless, by following the scheme of ref. 5 and by making a repeated use of (3) we will show that the metastable behavior of the system remains unchanged: the first problem to solve is the full characterization of the local minima of the hamiltonian. A local minimum is a configuration $\sigma \in \Omega$ such that for any nearest neighbor configuration η one has $H(\sigma) \leqslant H(\eta)$. Although the hamiltonian is the standard Ising hamiltonian, the notion of neighboring configurations is dramatically changed, so it is not obvious a priori that the local minima are the rectangles of pluses as in the single spin flip case. But this is the case, indeed we can prove that a configuration σ is a local minimum for the alternate dynamics if and only if it is a local minimum in the single spin flip case. It is easy to see that the sufficient condition holds because a pair of nearest neighboring configurations for the single spin flip dynamics is a pair of nearest neighboring configuration even in the alternate case. The necessity condition: let σ be a local minimum for the single spin flip dynamics and let η a e(o)-nn of σ. There exists $I = \{x_i\colon 1 \leqslant i \leqslant n\} \in \Xi^{e(o)}$ such that $\eta = \sigma^I$; the proof follows from (3) and from the fact that any σ^{x_i} is a nearest neighbor of σ in the single spin flip case.

The second step in the understanding of the metastable behavior of the model is the description of the tendency to shrink or to grow of the local minima. A preliminary definition: given $\eta \in \Omega$ if σ_t is the process started at some $\sigma_0 \in \Omega$ we denote by $\tau_\eta := \{t > 0 : \sigma_t = \eta\}$ the first hitting time on η. Now we state the following lemma on the criticality of rectangles: a rectangle $R_{\ell, m}$ is a configuration with a rectangular droplet of pluses, with side lengths ℓ and m, plunged in the sea of minuses.

Lemma 1. Let $\ell^* := [\frac{2J}{h}] + 1$; consider a rectangle $R_{\ell, m} \in \Omega$, with $\ell \leqslant m$, and the Markov chain σ_t started at $\sigma_0 = R_{\ell, m}$. Given $\varepsilon > 0$ we have: if $\ell < \ell^*$ then $R_{\ell,m}$ is subcritical, that is $P(\tau_{-\underline{1}} < \tau_{+\underline{1}}) \xrightarrow{\beta\to\infty} 1$, and $P(\exp\{\beta(\ell-1)\, h - \beta\varepsilon\} < \tau_{-\underline{1}} < \exp\{\beta(\ell-1)\, h + \beta\varepsilon\}) \xrightarrow{\beta\to\infty} 1$. If $\ell \geqslant \ell^*$ then $R_{\ell,m}$ is supercritical, that is $P(\tau_{+\underline{1}} < \tau_{-\underline{1}}) \xrightarrow{\beta\to\infty} 1$, and $P(\exp\{\beta(2J-h) - \beta\varepsilon\} < \tau_{+\underline{1}} < \exp\{\beta(2J-h) + \beta\varepsilon\}) \xrightarrow{\beta\to\infty} 1$.

To prove the lemma we need few more definitions: a downhill path is a sequence of configuration $\sigma_0, \sigma_1, ..., \sigma_n$ such that depending on i even or odd σ_i and σ_{i+1} are e(o)-nn and for any $i = 0,..., n-1$ $H(\sigma_i) \geqslant H(\sigma_{i+1})$; an uphill path is the reverse of a downhill one. The basin of attraction of a local minimum σ is the set $B(\sigma) := \{\eta \in \Omega :$ all the downhill paths starting from η end in $\sigma\}$, and for any $\mathscr{A} \subset \Omega$ its boundary $\partial\mathscr{A}$ is the collection of configurations $\eta \in \Omega \setminus \mathscr{A}$ such that there exists $\sigma \in \mathscr{A}$ nearest neighbor of η.

Now, we just have to find the minimum of the energy on $\partial B(R_{\ell,m})$ and use the results in Propositions 3.4 and 3.7 of ref. 5. Starting from $R_{\ell,m}$ one should consider all the possible uphill paths reaching $\partial B(R_{\ell,m})$, but (3) implies that all the mechanisms involving more than a single spin flip are energetically less favorable; hence the minimum of the energy on $\partial B(R_{\ell,m})$ can be found using the same arguments as in the single spin flip case. From this remark the proof of the lemma follows.

Now we state the theorem which describes the exit from the metastable phase: let us denote by $\mathscr{P}$ the configuration with all the spins minus excepted those in a $\ell^* \times (\ell^*-1)$ rectangle and in a unit protuberance attached to one of its longest sides. We set $\Gamma := H(\mathscr{P}) - H(-\underline{1}) = 4J\ell^* - h(\ell^{*2} - \ell^* + 1)$, we consider the process σ_t starting from $-\underline{1}$ and we define $\bar{\tau}_{-\underline{1}} := \sup\{t < \tau_{+\underline{1}} : \sigma_t = -\underline{1}\}$, the last time the system visits $-\underline{1}$ before reaching $+\underline{1}$, and $\bar{\tau}_{\mathscr{P}} := \inf\{t > \bar{\tau}_{-\underline{1}} : \sigma_t = \mathscr{P}\}$, the first time the system reaches $\mathscr{P}$ after having "definitively" left $-\underline{1}$. Finally, we can state the following

Theorem 1. Let us suppose $\sigma_0 = -\underline{1}$, let $\varepsilon > 0$, then $P(\bar{\tau}_{\mathscr{P}} < \tau_{+\underline{1}}) \xrightarrow{\beta\to\infty} 1$ and $P(\exp\{\beta\Gamma - \beta\varepsilon\} < \tau_{+\underline{1}} < \exp\{\beta\Gamma + \beta\varepsilon\} \xrightarrow{\beta\to\infty} 1$.

We sketch the proof of the theorem. We use the same notation as in Section 3.1 of ref. 6. First of all we consider the map $S: \Omega \to \Omega$ morally associating to each configuration $\sigma \in \Omega$ the "largest" local minimum $S(\sigma)$ to which σ is connected by a downhill path, where downhill must be understood in the sense of one spin nearest neighboring configurations (configurations differing for the value of a single spin). See Section 3.1 of ref. 6 for the rigorous definition of S. Now, we define the set $\mathscr{A} \subset \Omega$ as the collection of configurations $\sigma \in \Omega$ such that all the plus spins of $S(\sigma)$ are those associated to the sites inside a collection of rectangles on the lattice such that however two of such rectangles are chosen, their mutual distance is greater or equal to $\sqrt{5}$ (pairwaise non–interacting rectangles).

Following ref. 5, we prove that $H(\mathscr{P})$ is the minimum of the energy on the boundary of $\mathscr{A}$. Let us partition $\partial\mathscr{A}$ into two parts $\partial\mathscr{A} = \partial_1\mathscr{A} \cup \partial_{\geqslant 2}\mathscr{A}$: for each $\eta \in \partial\mathscr{A}$, $\eta \in \partial_1\mathscr{A}$ iff there exists $x \in \Lambda$ such that $\sigma^x \in \mathscr{A}$, while $\eta \in \partial_{\geqslant 2}\mathscr{A}$ iff $\sigma^x \in \Omega \setminus \mathscr{A}$ for all $x \in \Lambda$ and there exists $I \in \Xi^{e(o)}$ with $|I| \geqslant 2$ such that $\sigma^I \in \mathscr{A}$. We first prove that for all $\eta \in \partial_{\geqslant 2}\mathscr{A}$ there exists $\zeta \in \partial_1\mathscr{A}$ such that $H(\zeta) < H(\eta)$. Let $\eta \in \partial_{\geqslant 2}\mathscr{A}$ and $I \in \Xi^{e(o)}$ the subset of Λ with smallest cardinality such that $\sigma := \eta^I \in \mathscr{A}$. The definition of $\partial_{\geqslant 2}\mathscr{A}$ implies $|I| \geqslant 2$; the minimality of I and the construction of S imply $\eta(x) = +1$ and

$$H(\eta^x) < H(\eta) \qquad \text{for all} \quad x \in I. \tag{4}$$

Now, let $x \in I$, from the minimality of I we get $\zeta := \eta^{I \setminus \{x\}} \in \Omega \setminus \mathscr{A}$. Hence, $\zeta^x = \sigma \in \mathscr{A}$ and $\zeta \in \Omega \setminus \mathscr{A}$ imply $\zeta \in \partial_1\mathscr{A}$. Moreover, from $|I| \geqslant 2$, equation (3) and the property (4) we, finally, get $H(\zeta) < H(\eta)$.

Now, by using the same strategy as in Section 3.1 of ref. 6 we get that the minimum of the energy on $\partial\mathscr{A}$ is realized in $\mathscr{P}$. Finally, Theorem 1 is a consequence of Propositions 3.4 and 3.7 of ref. 5.

In conclusion we can say that in the case of the alternate updating the property (3) implies that the mechanism of escape from the metastable phase is not changed with respect to the Glauber case; on the other hand one can expect that things could be different if the parallel updating were allowed all over the lattice without any restriction.[7]

ACKNOWLEDGMENTS

E.C. wishes to express his thanks to J. L. Lebowitz for having suggested the problem and for having carefully read the manuscript. E.C. also thanks F. R. Nardi and E. Olivieri for very useful discussions and the European network "Stochastic Analysis and its Applications" ERB-FMRX-CT96-0075 for financial support.

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